ALGEBRAIC AND TOPOLOGICAL PROPERTIES OF SOME SETS IN ℓ_1

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Abstract. For a sequence $x \in \ell_1 \setminus c_{00}$, one can consider the set $E(x)$ of all subsums of series $\sum_{n=1}^{\infty} x(n)$. Guthrie and Nymann proved that $E(x)$ is one of the following types of sets:

($\mathcal{I}$) a finite union of closed intervals;

($\mathcal{C}$) homeomorphic to the Cantor set;

($\mathcal{MC}$) homeomorphic to the set T of subsums of $\sum_{n=1}^{\infty} b(n)$ where $b(2n-1) = 3/4^n$ and $b(2n) = 2/4^n$.

By $\mathcal{I}$, $\mathcal{C}$ and $\mathcal{MC}$ denote the sets of all sequences $x \in \ell_1 \setminus c_{00}$, such that $E(x)$ has the property ($\mathcal{I}$), ($\mathcal{C}$) and ($\mathcal{MC}$), respectively. In this note we show that $\mathcal{I}$ and $\mathcal{C}$ are strongly $\mathfrak{c}$-algebrable and $\mathcal{MC}$ is $\mathfrak{c}$-lineable. We show that $\mathcal{C}$ is a dense G_δ-set in ℓ_1 and $\mathcal{I}$ is a true $\mathcal{F}_\sigma$-set. Finally we show that $\mathcal{I}$ is spaceable while $\mathcal{C}$ is not spaceable.

1. Introduction

1.1. **Lineability, algebrability and spaceability.** Having a linear algebra A and its subset $E \subset A$ one can ask if $E \cup \{0\}$ contains a linear subalgebra A' of A. Roughly speaking if the answer is positive, then E is algebrable. It is a recent trend in Mathematical Analysis to establish the algebrability of sets E which are far from being linear, that is $x, y \in E$ does not generally imply $x + y \in E$. Such algebrability results were obtained in sequence spaces (see [7], [6] and [8]) and in function spaces (see [2], [5], [4], [12] and [13]).

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Assume that V is a linear space (linear algebra). A subset $E \subset V$ is called lineable (algebrable) whenever $E \cup \{0\}$ contains an infinite-dimensional linear space (infinitely generated linear algebra, respectively), see [3], [9] and [15]. For a cardinal $\kappa > \omega$, let us observe that the set E is κ-algebrable (i.e. it contains κ-generated linear algebra), if and only if it contains an algebra which is a κ-dimensional linear space (see [7]). Moreover, we say that a subset E of a commutative linear algebra V is strongly κ-algebrable ([7]), if there exists a κ-generated free algebra A contained in $E \cup \{0\}$.

Note, that $X = \{x_\alpha : \alpha < \kappa\} \subset E$ is a set of free generators of a free algebra $A \subset E$ if and only if the set X' of elements of the form $x_{\alpha_1}^{k_1} x_{\alpha_2}^{k_2} \dots x_{\alpha_n}^{k_n}$ is linearly independent and all linear combinations of elements from X' are in $E \cup \{0\}$. It is easy to see that free algebras have no divisors of zero.

In practice, to prove κ-algebrability of set $E \subset V$ we have to find $X \subseteq E$ of cardinality κ such that for any polynomial P in n variables and any distinct $x_1, \dots, x_n \in X$ we have either $P(x_1, \dots, x_n) \in E$ or $P(x_1, \dots, x_n) = 0$. To prove the strong κ-algebrability of E we have to find $X \subset E$, $|X| = \kappa$, such that for any non-zero polynomial P and distinct $x_1, \dots, x_n \in X$ we have $P(x_1, \dots, x_n) \in E$.

In general, there are subsets of linear algebras which are algebrable but not strongly algebrable. Let c_{00} be a subset of c_0 consisting of all sequences with real terms equal to zero from some place. Then the set c_{00} is algebrable in c_0 but is not strongly 1-algebrable [7].

Let X be a Banach space. The subset M of X is spaceable if $M \cup \{0\}$ contains infinitely dimensional closed subspace Y of X. Since every infinitely dimensional Banach space contains linearly independent set of the cardinality continuum, the spaceability implies $\mathfrak{c}$-lineability. However, the spaceability is a much stronger property then $\mathfrak{c}$-lineability. The notions of spaceability and $\mathfrak{c}$-algebrability are incomparable. We will show that even $\mathfrak{c}$-algebrable dense $\mathcal{G}_\delta$-sets in ℓ_1 may not be spaceable. On the other hand, there are sets in c_0 which are spaceable but not 1-algebrable (see [7]).

1.2. **The subsums of series.** Let $x \in \ell_1$. The set of all subsums of $\sum_{n=1}^{\infty} x(n)$, meaning the set of sums of all subseries of $\sum_{n=1}^{\infty} x(n)$, is defined by

$$E(x) = \{a \in \mathbb{R} : \exists A \subset \mathbb{N} \quad \sum_{n \in A} x(n) = a\}.$$

Some authors call it the achievement set of x. The following theorem is due to Kakeya.

Theorem 1. [18]. *Let $x \in \ell_1$*

(1) *If $x \notin c_{00}$, then $E(x)$ is a perfect compact set.*

(2) *If $|x(n)| > \sum_{i>n} |x(i)|$ for almost all n, then $E(x)$ is homeomorphic to the Cantor set.*

(3) *If $|x(n)| \leq \sum_{i>n} |x(i)|$ for n sufficiently large, then $E(x)$ is a finite union of closed intervals. In the case of non-increasing sequence x, the last inequality is also necessary to obtain $E(x)$ being a finite union of intervals.*

Moreover, Kakeya conjectured that $E(x)$ is either nowhere dense or it is a finite union of intervals. Probably, the first counterexample to this conjecture was given (without a proof) by Weinstein and Shapiro [21] and, with a correct proof, by Ferens [11]. Guthrie and Nymann [16] showed that, for the sequence b given by the formulas $b(2n-1) = \frac{3}{4^n}$ and $b(2n) = \frac{2}{4^n}$, the set $T = E(b)$ is not a finite union of intervals but it has nonempty interior. In the same paper they formulated the following theorem

Theorem 2. [16] *Let $x \in \ell_1 \setminus c_{00}$, then $E(x)$ is one of the following sets:*

(i) *a finite union of closed intervals;*

(ii) *homeomorphic to the Cantor set;*

(iii) *homeomorphic to the set T.*

A correct proof of the Guthrie and Nymann trichotomy was given by Nymann and Sáenz [20]. The sets homeomorphic to T are called Cantorvals (more precisely: M-Cantorvals). Note that Theorem 2 can be formulated as follows: The space ℓ_1 is a disjoint union of the sets $c_{00}, \mathcal{I}, \mathcal{C}$ and $\mathcal{MC}$ where

$\mathcal{I}$ consists of sequences x with $E(x)$ equal to a finite union of intervals, $\mathcal{C}$ consists of sequences x with $E(x)$ homeomorphic to the Cantor set, and $\mathcal{MC}$ of x with $E(x)$ being an M-Cantorval.

For $x \in \ell_1$, let x' be an arbitrary finite modification of x, and let $|x|$ denote the sequence $y \in \ell_1$ such that $y(n) = |x(n)|$. Then $x \in \mathcal{I} \iff |x| \in \mathcal{I} \iff x' \in \mathcal{I}$. The same equivalences hold for sets $\mathcal{C}$ and $\mathcal{MC}$.

2. Algebraic substructures in $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{MC}$.

Jones in a very nice paper [17] gives the following example. Let $x(n) = 1/2^n$ and $y(n) = 1/3^n$. Then clearly $x \in \mathcal{I}$ and $y \in \mathcal{C}$. Moreover, $x + y \in \mathcal{C}$ and $x - y \in \mathcal{I}$. Since $x = (x + y) - y$ and $y = -(x - y) + x$, then neither $\mathcal{I}$ nor $\mathcal{C}$ is closed under pointwise addition. However, in the present paper we show that the sets $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{MC}$ contain large ($\mathfrak{c}$-generated) algebraic structures. To prove the strong $\mathfrak{c}$-algebrability of $\mathcal{C}$ and $\mathcal{I}$, we will combine Theorem 1 and the method of linearly independent exponents, which was successful in [6] and [7]. In the next theorem we construct generators as the powers of one geometric series x_q ($x_q(n) = q^n$) for $0 < q < \frac{1}{2}$. Clearly, by Theorem 1, $x_q \in \mathcal{C}$.

Theorem 3. *$\mathcal{C}$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Fix $q \in (0, 1/2)$. Let $\{r_\alpha : \alpha < \mathfrak{c}\}$ be a linearly independent (over the field of all rationals $\mathbb{Q}$) set of reals greater than 1. Let $x_\alpha(n) = q^{r_\alpha n}$. We will show that the set $\{x_\alpha : \alpha < \mathfrak{c}\}$ generates a free algebra $\mathcal{A}$ which, except for the null sequence, is contained in $\mathcal{C}$.

To do this, we will show that for any $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R} \setminus \{0\}$, any matrix $[k_{il}]_{i \le m, l \le j}$ of natural numbers with nonzero distinct rows, and any $\alpha_1 < \alpha_2 < \cdots < \alpha_j < \mathfrak{c}$, the sequence x given by

$$x(n) = P(x_{\alpha_1}, \ldots, x_{\alpha_j})(n)$$

where

$$P(z_1,\dots,z_j)=\beta_1 z_1^{k_{11}}z_2^{k_{12}}\dots z_j^{k_{1j}}+\dots+\beta_m z_1^{k_{m1}}z_2^{k_{m2}}\dots z_j^{k_{mj}}$$

is in $\mathcal{C}$. In other words,

$$x(n)=\beta_1 q^{n(r_{\alpha_1}k_{11}+\dots+r_{\alpha_j}k_{1j})}+\dots+\beta_m q^{n(r_{\alpha_1}k_{m1}+\dots+r_{\alpha_j}k_{mj})}$$

Since $r_{\alpha_1},\dots,r_{\alpha_j}$ are linearly independent and the rows of $[k_{il}]_{i\leqslant m,l\leqslant j}$ are distinct, the numbers $r_1:=r_{\alpha_1}k_{11}+\dots+r_{\alpha_j}k_{1j},\dots,r_m:=r_{\alpha_1}k_{m1}+\dots+r_{\alpha_j}k_{mj}$ are distinct. We may assume that $r_1<\dots<r_m$. Then

$$\frac{|x(n)|}{\sum_{i>n}|x(i)|}=\frac{|\beta_1 q^{nr_1}+\dots+\beta_m q^{nr_m}|}{\sum_{i>n}|\beta_1 q^{ir_1}+\dots+\beta_m q^{ir_m}|}$$

$$\geq\frac{|\beta_1 q^{nr_1}+\dots+\beta_m q^{nr_m}|}{\sum_{i>n}(|\beta_1|q^{ir_1}+\dots+|\beta_m|q^{ir_m})}=\frac{|\beta_1 q^{nr_1}+\dots+\beta_m q^{nr_m}|}{\frac{|\beta_1|q^{(n+1)r_1}}{1-q^{r_1}}+\dots+\frac{|\beta_m|q^{(n+1)r_m}}{1-q^{r_m}}}$$

$$\to\frac{1-q^{r_1}}{q^{r_1}}>1.$$

Therefore there is n_0, such that $|x(n)|>\sum_{i>n}|x(i)|$ for all $n\geq n_0$. Hence, by Theorem 1, we obtain that $x\in\mathcal{C}$.

□

It is obvious that the geometric sequence x_q, even for $q>\frac{1}{2}$, is not useful to construct the generators of linear algebra contained in $\mathcal{I}$. Indeed, for sufficiently large exponent k, the sequence x_q^k belongs to $\mathcal{C}$. So, in the next theorem we use the harmonic series.

Theorem 4. *$\mathcal{I}$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Let K be a linearly independent subset of $(1,\infty)$ of cardinality $\mathfrak{c}$. For $\alpha\in K$, let x_α be a sequence given by the formula $x_\alpha(n)=\frac{1}{n^\alpha}$. We will show that the set $\{x_\alpha:\alpha\in K\}$ generates a free algebra $\mathcal{A}$ which is contained in $\mathcal{I}\cup\{0\}$. To do this, we will show that for any $\beta_1,\beta_2,\dots,\beta_m\in\mathbb{R}\setminus\{0\}$, any matrix $[k_{il}]_{i\leq m,l\leq j}$ of natural numbers with nonzero distinct rows, and any $\alpha_1<\alpha_2<\dots<\alpha_j$, the sequence x defined by

$$x=P(x_{\alpha_1},x_{\alpha_2},\dots,x_{\alpha_j})$$

$$= \beta_1 x_{\alpha_1}^{k_{11}} x_{\alpha_2}^{k_{12}} \dots x_{\alpha_j}^{k_{1j}} + \beta_2 x_{\alpha_1}^{k_{21}} x_{\alpha_2}^{k_{22}} \dots x_{\alpha_j}^{k_{2j}} + \dots + \beta_m x_{\alpha_1}^{k_{m1}} x_{\alpha_2}^{k_{m2}} \dots x_{\alpha_j}^{k_{mj}}$$

belongs to $\mathcal{I}$. We have

$$x(n) = P(x_{\alpha_1}, x_{\alpha_2}, \dots, x_{\alpha_j})(n)$$
$$= \beta_1 \frac{1}{n^{\alpha_1 k_{11} + \alpha_2 k_{12} + \dots + \alpha_j k_{1j}}} + \dots + \beta_m \frac{1}{n^{\alpha_1 k_{m1} + \alpha_2 k_{m2} + \dots + \alpha_j k_{mj}}}$$
$$= \beta_1 \frac{1}{n^{p_1}} + \beta_2 \frac{1}{n^{p_2}} + \dots + \beta_j \frac{1}{n^{p_m}}$$

Note that $p_1, \dots, p_m$ are distinct. Assume that $p_1 < p_2 < \dots < p_m$. We have

$$\frac{|x(n)|}{\sum_{k>n} |x(k)|} = \frac{|\beta_1 \frac{1}{n^{p_1}} + \beta_2 \frac{1}{n^{p_2}} + \dots + \beta_m \frac{1}{n^{p_m}}|}{\sum_{k>n} |\beta_1 \frac{1}{k^{p_1}} + \beta_2 \frac{1}{k^{p_2}} + \dots + \beta_m \frac{1}{k^{p_m}}|}$$
$$\leqslant \frac{|\beta_1 \frac{1}{n^{p_1}} + \beta_2 \frac{1}{n^{p_2}} + \dots + \beta_m \frac{1}{n^{p_m}}|}{\sum_{k>n} \left(|\beta_1 \frac{1}{k^{p_1}}| - |\beta_2 \frac{1}{k^{p_2}}| - \dots - |\beta_m \frac{1}{k^{p_m}}| \right)}$$
$$\leqslant \frac{|\beta_1 \frac{1}{n^{p_1}} + \beta_2 \frac{1}{n^{p_2}} + \dots + \beta_m \frac{1}{n^{p_m}}|}{|\beta_1| \int_{n+1}^{\infty} \frac{1}{x^{p_1}} dx - |\beta_2| \int_{n}^{\infty} \frac{1}{x^{p_2}} dx - \dots - |\beta_m| \int_{n}^{\infty} \frac{1}{x^{p_m}} dx}$$
$$= \frac{|\beta_1 + \beta_2 \frac{n^{p_1}}{n^{p_2}} + \dots + \beta_m \frac{n^{p_1}}{n^{p_m}}|}{n[|\beta_1| \frac{1}{p_1 - 1} \frac{n^{p_1 - 1}}{(n+1)^{p_1 - 1}} - |\beta_2| \frac{1}{p_2 - 1} \frac{n^{p_1 - 1}}{(n)^{p_2 - 1}} - \dots - |\beta_m| \frac{1}{p_m - 1} \frac{n^{p_1 - 1}}{(n)^{p_m - 1}}]}$$
$$\xrightarrow[n \to \infty]{} 0 < 1.$$

Observe that the first inequality holds for n large enough. Therefore there is n_0 such that $|x(n)| \leq \sum_{i>n} |x(i)|$ for any $n \geq n_0$. Hence, by Theorem 1 we obtain that $x \in \mathcal{I}$.

□

The method described in the next lemma belongs to the mathematical folklore and was used to construct sequences x's with $E(x)$ being Cantorvals. We present its proof since we did not find it explicitly formulated in the mathematical literature.

Lemma 5. *Let $x \in \ell_1$ be such that*

(i) *$E(x)$ contains an interval;*
(ii) *$|x(n)| > \sum_{i>n} |x(i)|$ for infinitely many n;*
(iii) *$|x_n| \geqslant |x_{n+1}|$ for almost all n.*

Then $x \in \mathcal{MC}$.

Proof. By (ii)-(iii), the point x does not belong to $\mathcal{I}$. By (i), the point x does not belong to $\mathcal{C}$. Hence, by Theorem 2 we get $x \in \mathcal{MC}$. □

Up to last years, there were only known a few examples of sequences belonging to $\mathcal{MC}$. These examples were not very useful to construct a large number of linearly independent sequences. Recently, Jones in [17] has constructed a one-parameter family of sequences in $\mathcal{MC}$. We shall use some modification of the example given by Jones in the proof of our next theorem.

Theorem 6. $\mathcal{MC}$ *is* $\mathfrak{c}$*-lineable.*

Proof. Let

$$x_q = (4, 3, 2, 4q, 3q, 2q, 4q^2, 3q^2, 2q^2, 4q^3, \dots)$$

and

$$y_q = (1, 1, 1, 1, 1, q, q, q, q, q, q^2, q^2, q^2, q^2, q^2, q^3, \dots)$$

for $q \in [\frac{1}{6}, \frac{2}{11})$.

Observe that the sequences $x_q, q \in [\frac{1}{6}, \frac{2}{11})$ are linearly independent. We need to show that each non-zero linear combination of sequences x_q fulfils the assumptions (i)–(iii) of Lemma 5 and therefore it is actually in $\mathcal{MC}$. To prove this, let us fix $q_1 > q_2 > \cdots > q_m \in [\frac{1}{6}, \frac{2}{11})$, $\beta_1, \beta_2, \dots, \beta_m \in \mathbb{R}$ and define sequences x and y by

$$x(n) = \beta_1 x_{q_1}(n) + \beta_2 x_{q_2}(n) + \cdots + \beta_m x_{q_m}(n)$$

and

$$y(n) = \beta_1 y_{q_1}(n) + \beta_2 y_{q_2}(n) + \cdots + \beta_m y_{q_m}(n).$$

At first, we will check that for almost all n

$$(1)\quad 2|\beta_1 {q_1}^n + \beta_2 {q_2}^n + \cdots + \beta_m {q_m}^n| > 9 \sum_{k>n} |\beta_1 {q_1}^k + \beta_2 {q_2}^k + \cdots + \beta_m {q_m}^k|.$$

We have

$$\frac{2|\beta_1 {q_1}^n + \beta_2 {q_2}^n + \cdots + \beta_m {q_m}^n|}{9\sum_{k>n} |\beta_1 {q_1}^k + \beta_2 {q_2}^k + \cdots + \beta_m {q_m}^k|} \geqslant \frac{2|\beta_1 {q_1}^n + \beta_2 {q_2}^n + \cdots + \beta_m {q_m}^n|}{9\sum_{k>n} |\beta_1 {q_1}^k| + |\beta_2 {q_2}^k| + \cdots + |\beta_m {q_m}^k|}$$

$$= \frac{2|\beta_1 {q_1}^n + \beta_2 {q_2}^n + \cdots + \beta_m {q_m}^n|}{9(|\beta_1|\frac{{q_1}^{n+1}}{1-q_1} + |\beta_2|\frac{{q_2}^{n+1}}{1-q_2} + \cdots + |\beta_m|\frac{{q_m}^{n+1}}{1-q_m})} \underset{n\to\infty}{\longrightarrow} \frac{2}{9}\cdot\frac{1-q_1}{q_1} > \frac{2}{9}\cdot\frac{1-\frac{2}{11}}{\frac{2}{11}} = 1.$$

Note that if n is not divisible by 3, then $|x(n)| \geqslant |x(n+1)|$. On the other hand, if $n = 3l$, then

$$|x(n)| = 2|\beta_1 q_1^l + \cdots + \beta_m q_m^l|$$

and

$$|x(n+1)| = 3|\beta_1 q_1^{l+1} + \cdots + \beta_m q_m^{l+1})| \leqslant 9\sum_{k>l} |\beta_1 q_1^k + \cdots + \beta_m q_m^k|.$$

Hence by (1) we obtain $|x(n)| \geqslant |x(n+1)|$ for almost all n. By (1) we also have $|x(n)| > \sum_{i>n} |x(i)|$ for infinitely many n.

Now we will show that

$$(2) \quad |\beta_1 {q_1}^n + \beta_2 {q_2}^n + \cdots + \beta_m {q_m}^n| \leqslant 5\sum_{k>n} |\beta_1 {q_1}^k + \beta_2 {q_2}^k + \cdots + \beta_m {q_m}^k|.$$

We have

$$\frac{|\beta_1 {q_1}^n + \beta_2 {q_2}^n + \cdots + \beta_m {q_m}^n|}{5\sum_{k>n} |\beta_1 {q_1}^k + \beta_2 {q_2}^k + \cdots + \beta_m {q_m}^k|} \leqslant \frac{|\beta_1 {q_1}^n + \beta_2 {q_2}^n + \cdots + \beta_m {q_m}^n|}{5|\sum_{k>n} \beta_1 {q_1}^k + \beta_2 {q_2}^k + \cdots + \beta_m {q_m}^k|}$$

$$= \frac{|\beta_1 + \beta_2(\frac{q_2}{q_1})^n + \cdots + \beta_m(\frac{q_m}{q_1})^n|}{5|\beta_1 \sum_{i>0} {q_1}^i + \beta_2(\frac{q_2}{q_1})^n \sum_{i>0} {q_2}^i + \cdots + \beta_m(\frac{q_m}{q_1})^n \sum_{i>0} {q_m}^i|}$$

$$\underset{n\to\infty}{\longrightarrow} \frac{1}{5}\cdot\frac{1-q_1}{q_1} \leqslant \frac{1}{5}\cdot\frac{1-\frac{1}{6}}{\frac{1}{6}} = 1.$$

By (2) we obtain that $|y(n)| \leqslant \sum_{k>n} |y(k)|$ for almost all n. Therefore by Theorem 1, the set $E(y)$ is a finite union of closed intervals. Thus $E(y)$ has non-empty interior.

To end the proof we need to show that $E(x)$ has non-empty interior. We will prove that

$$2\sum_{n=0} (\beta_1 {q_1}^n + \beta_2 {q_2}^n + \cdots + \beta_m {q_m}^n) + E(y) \subseteq E(x).$$

Let

$$t \in 2\sum_{n=0} (\beta_1 {q_1}^n + \beta_2 {q_2}^n + \cdots + \beta_m {q_m}^n) + E(y).$$

Note that any element t of $E(y)$ is of the form

$$t = k_0(\beta_1 + \beta_2 + \dots + \beta_m) + k_1(\beta_1 q_1 + \beta_2 q_2 + \dots + \beta_m q_m)$$

$$+k_2(\beta_1 {q_1}^2 + \beta_2 {q_2}^2 + \dots + \beta_m {q_m}^2) + \dots$$

where $k_n \in \{0, 1, 2, 3, 4, 5\}$. Thus t is of the form

$$t = 2\sum_{n=0}(\beta_1 {q_1}^n + \beta_2 {q_2}^n + \dots + \beta_m {q_m}^n) +$$

$$+[k_0(\beta_1 + \beta_2 + \dots + \beta_m) + k_1(\beta_1 q_1 + \beta_2 q_2 + \dots + \beta_m q_m)$$

$$+k_2(\beta_1 {q_1}^2 + \beta_2 {q_2}^2 + \dots + \beta_m {q_m}^2) + \dots]$$

$$= (2 + k_0)(\beta_1 + \beta_2 + \dots + \beta_m) + (2 + k_1)(\beta_1 q_1 + \beta_2 q_2 + \dots + \beta_m q_m) +$$

$$+(2 + k_2)(\beta_1 {q_1}^2 + \beta_2 {q_2}^2 + \dots + \beta_m {q_m}^2) + \dots$$

Note that each number from $\{2, 3, 4, 5, 6, 7\}$, that is every number of the form $2+k_n$, can be written as a sum of numbers $4, 3, 2$. Hence $t \in E(x)$ and $E(x)$ has non-empty interior. So $x \in \mathcal{MC}$. □

3. The topological size and Borel classification of $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{MC}$.

Let us observe that all the sets $c_{00}, \mathcal{C}, \mathcal{I}$ and $\mathcal{MC}$ are dense in ℓ_1. Moreover, c_{00} is an $\mathcal{F}_\sigma$-set of the first category. We are interested in studying the topological size and Borel classification of considered sets. To do it, let us consider the hyperspace $H(\mathbb{R})$, that is the space of all non-empty compact subsets of reals, equipped with the Vietoris topology (see [19], 4F, pp.24-28). Recall, that the Vietoris topology is generated by the subbase of sets of the form $\{K \in H(\mathbb{R}) : K \subset U\}$ and $\{K \in H(\mathbb{R}) : K \cap U \neq \emptyset\}$ for all open sets U in $\mathbb{R}$. This topology is metrizable by the Hausdorff metric d_H given by the formula

$$d_H(A, B) = \max\{\max_{t\in A} d(t, B), \max_{s\in B} d(s, A)\}$$

where d is the natural metric in $\mathbb{R}$. It is known that the set N of all nowhere dense compact sets is a G_δ-set in $H(\mathbb{R})$ and the set F of all compact sets with finite number of connected components is an $\mathcal{F}_\sigma$-set. To see this, it is enough to observe that

- K is nowhere dense if and only if for any set U_n from a fixed countable base of natural topology in $\mathbb{R}$ there exists a set U_m from this base, such that $cl(U_m) \subset U_n$ and $K \subset (cl(U_m))^c$;
- K has more then k components if and only if there exist pairwise disjoint open intervals $J_1, J_2, \ldots, J_{k+1}$, such that $K \subset J_1 \cup J_2 \cup \cdots \cup J_{k+1}$ and $K \cap J_i \neq \emptyset$ for $i = 1, 2, \ldots, k+1$.

Now, let us observe that if we assign the set $E(x)$ to the sequence $x \in \ell_1$, we actually define the function $E : \ell_1 \to H(\mathbb{R})$.

Lemma 7. *The function E is Lipschitz with Lipschitz constant $L = 1$, hence it is continuous.*

Proof. Let $t \in E(x)$. Then there exists a subset A of $\mathbb{N}$ such that $t = \sum_{n\in A} x(n)$. We have

$$d(t, E(y)) \leqslant d(t, \sum_{n\in A} y(n)) = \left|\sum_{n\in A}(x(n) - y(n))\right| \leqslant \sum_{n\in\mathbb{N}} |(x(n)-y(n))| = \|x-y\|_1$$

where $\|\cdot\|_1$ denotes the norm in ℓ_1. Hence, $d_H(E(x), E(y)) \leqslant \|x - y\|_1$. □

Theorem 8. *The set $\mathcal{C}$ is a dense G_δ-set (and hence residual), $\mathcal{I}$ is a true $\mathcal{F}_\sigma$-set (i.e. it is $\mathcal{F}_\sigma$ but not $\mathcal{G}_\delta$) of the first category, and $\mathcal{MC}$ is in the class $(\mathcal{F}_{\sigma\delta} \cap \mathcal{G}_{\delta\sigma}) \setminus \mathcal{G}_\delta$.*

Proof. Let us observe that $\mathcal{C} \cup c_{00} = E^{-1}[N]$ and $\mathcal{I} \cup c_{00} = E^{-1}[F]$ where N, F, E are defined as before. Hence $\mathcal{C} \cup c_{00}$ is G_δ-set and $\mathcal{I} \cup c_{00}$ is $\mathcal{F}_\sigma$-set. Thus $\mathcal{C}$ is G_δ-set (because c_{00} is $\mathcal{F}_\sigma$-set) and $\mathcal{I} \cup \mathcal{MC}$ is $\mathcal{F}_\sigma$. Moreover, $\mathcal{I} = (\mathcal{I} \cup c_{00}) \cap (\mathcal{I} \cup \mathcal{MC})$ is $\mathcal{F}_\sigma$-set, too. By the density of $\mathcal{C}$, $\mathcal{C}$ is residual. Since $\mathcal{I}$ is dense of the first category, it cannot be $\mathcal{G}_\delta$-set. For the same reason, $\mathcal{MC}$ also cannot be $\mathcal{G}_\delta$-set. Since $\mathcal{MC}$ is a difference of two $\mathcal{F}_\sigma$-sets, it is in the class $\mathcal{F}_{\sigma\delta} \cap \mathcal{G}_{\delta\sigma}$. □

Remark 9. In [7] it was shown the following similar result by the use of quite different methods: the set of bounded sequences, with the set of limit points homeomorphic to the Cantor set, is strongly $\mathfrak{c}$-algebrable and residual in l^∞.

4. Spaceability

In this section we will show that $\mathcal{I}$ is spaceable while $\mathcal{C}$ is not spaceable. This shows that there is a subset M of ℓ_1 containing a dense $\mathcal{G}_\delta$ subset and such that it contains a linear subspace of dimension $\mathfrak{c}$, but $Y \setminus M \neq \emptyset$ for any infinitely dimensional closed subspace Y of ℓ_1.

Theorem 10. *Let $\mathcal{I}_1$ be a subset of $\mathcal{I}$ which consists of those $x \in \ell_1$ for which $E(x)$ is an interval. Then $\mathcal{I}_1$ is spaceable.*

Proof. Let $A_1, A_2, \dots$ be a partition of $\mathbb{N}$ into infinitely many infinite subsets. Let $A_n = \{k_n^1 < k_n^2 < k_n^3 < \dots\}$. Define $x_n \in \ell_1$ in the following way. Let $x_n(k_n^j) = 2^{-j}$ and $x_n(i) = 0$ if $i \notin A_n$. Then $\|x_n\|_1 = 1$ and $\{x_n : x \in \mathbb{N}\}$ forms a normalised basic sequence. Let Y be a closed linear space generated by $\{x_n : x \in \mathbb{N}\}$. Then

$$y \in Y \iff \exists t \in \ell_1 \Big(y = \sum_{n=1}^{\infty} t(n) x_n \Big).$$

Since $E(x_n) = [0,1]$, then $E(\sum_{n=1}^{\infty} t(n)x_n) = \bigcup_{n=1}^{\infty} I_n$ where I_n is an interval with endpoints 0 and $t(n)$. Put $t^+(n) = \max\{t(n), 0\}$ and $t^-(n) = \min\{-t(n), 0\}$. Then $E(\sum_{n=1}^{\infty} t(n)x_n) = [\sum_{n=1}^{\infty} t^-(n), \sum_{n=1}^{\infty} t^+(n)]$ and the result follows. □

Let us remark the very recent result by Bernal-González and Ordóñez Cabrera [10, Theorem 2.2]. The authors gave sufficient conditions for spaceability of sets in Banach spaces. Using that result, one can prove spaceability of $\mathcal{I}$ but it cannot be used to prove Theorem 10, since the assumptions are not fulfilled.

However we do not know more results giving the sufficient conditions for a set in Banach space to not be spaceable. An interesting example of a non-spaceable set was given in the classical paper [14] by Gurarii where it was proved that the set of all differentiable functions from $C[0,1]$ is not spaceable. It is well known that the set of all differentiable functions in $C[0,1]$ is dense but meager. We will prove that even dense $\mathcal{G}_\delta$-sets in Banach spaces may not be spaceable.

Theorem 11. *Let Y be an infinitely dimensional closed subspace of ℓ_1. Then there is $y \in Y$ such that $E(y)$ contains an interval.*

Proof. Let Y be an infinitely dimensional closed subspace of ℓ_1. Let $\varepsilon_n \searrow 0$. Let x_1 be any nonzero element of Y with $\|x_1\|_1 = 1+\varepsilon_1$. Since $x_1 \in \ell_1$, there is n_1 with $\sum_{n=n_1+1}^\infty |x_1(n)| \leq \varepsilon_1$. Let E_1 consist of finite sums $\sum_{n=1}^{n_1} \delta_n x_1(n)$ where $\delta_i \in \{0,1\}$. Then E_1 is a finite set with $\min E_1 = \sum_{n=1}^{n_1} x_1^-(n)$, $\max E_1 = \sum_{n=1}^{n_1} x_1^+(n)$ and $1 \leq \max E_1 - \min E_1 \leq 1+\varepsilon_1$.

Let $Y_1 = Y \cap \{x \in \ell_1 : x(n) = 0 \text{ for every } n \leq n_1\}$. Since $\{x \in \ell_1 : x(n) = 0 \text{ for every } n \leq n_1\}$ has a finite co-dimension, then Y_1 is infinitely dimensional. Let x_2 be any nonzero element of Y_1 with $\|x_2\|_1 = 1+\varepsilon_2$. Since $x_2 \in \ell_1$, there is $n_2 > n_1$ with $\sum_{n=n_2+1}^\infty |x_i(n)| \leq \varepsilon_2$, $i = 1,2$. Let E_2 consist of finite sums $\sum_{n=n_1+1}^{n_2} \delta_n x_2(n)$, where $\delta_i \in \{0,1\}$. Then E_2 is a finite set with $\min E_2 = \sum_{n=n_1+1}^{n_2} x_2^-(n)$, $\max E_2 = \sum_{n=n_1+1}^{n_2} x_2^+(n)$ and $1 \leq \max E_2 - \min E_2 \leq 1+\varepsilon_2$.

Proceeding inductively, we define natural numbers $n_1 < n_2 < n_3 < \dots$, infinitely dimensional closed spaces $Y \supset Y_1 \supset Y_2 \supset \dots$ such that $Y_k = \{x \in Y : x(n) = 0 \text{ for every } n \leq n_k\}$, nonzero elements $x_k \in Y_{k-1}$ with $\|x_k\|_1 = 1+\varepsilon_k$ and $\sum_{n=n_k+1}^\infty |x_i(n)| \leq \varepsilon_k$, $i = 1,2,\dots,k$, and finite sets E_k consisting of sums $\sum_{n=n_{k-1}+1}^{n_k} \delta_n x_k(n)$ where $\delta_i \in \{0,1\}$. Note that $1 \leq \operatorname{diam}(E_k) \leq 1+\varepsilon_k$. Consider $y = \sum_{k=1}^\infty x_k/2^k$. We claim that $E(y)$ contains an interval $I := [\min E_1, \max E_1]$.

Note that for any $t \in I$ there is $t_1 \in E_1$ with $|t - t_1| \leq (1+\varepsilon_1)/2$. Since $1 \leq \operatorname{diam}(E_2) \leq 1 + \varepsilon_2$, there is $t_2 \in E_1 + \frac{1}{2}E_2$ with $|t - t_2| \leq (1+\varepsilon_2)/2^2$. Hence, there is $\tilde{t} \in E(x_1 + x_2/2)$ with $|t - \tilde{t}| \leq (1+\varepsilon_2)/2^2 + \varepsilon_1$. Since $1 \leq \operatorname{diam}(E_k) \leq 1+\varepsilon_k$, then inductively we can find $t_k \in E_1 + \frac{1}{2}E_2 + \cdots + \frac{1}{2^{k-1}}E_k$ with $|t - t_k| \leq (1+\varepsilon_k)/2^k$. Hence, there is $\tilde{t} \in E(x_1 + x_2/2 + \cdots + x_k/2^{k-1})$ with $|t-\tilde{t}| \leq (1+\varepsilon_k)/2^k + \varepsilon_{k-1} + \varepsilon_{k-1}/2 + \cdots + \varepsilon_{k-1}/2^{k-1} \leq (1+\varepsilon_k)/2^k + 2\varepsilon_{k-1}$. Since $E(y)$ is closed and it contains $E(x_1 + x_2/2 + \cdots + x_k/2^{k-1})$, then $t \in E(y)$ and consequently $I \subset E(y)$. □

Immediately we get the following.

Corollary 12. *The set $\mathcal{C}$ is not spaceable.*

We end the paper with the list of open questions on the set $\mathcal{MC}$.

Problem 13. (i) *Is $\mathcal{MC}$ $\mathfrak{c}$-algebrable?*

(ii) *Is $\mathcal{MC}$ an $\mathcal{F}_\sigma$ subset of ℓ_1?*

(iii) *Is $\mathcal{MC}$ spaceable?*

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